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Reply to Comments on ''Simple measure for complexity''

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We respond to the comment by Crutchfield, Feldman, and Shalizi [Comment in this issue, Phys. Rev. E **62**, 2996 (2000)] and that by Binder and Perry [preceding Comment, Phys. Rev. E **62**, 2998 (2000)], pointing out that there may be many maximum entropies, and therefore ''disorders'' and ''simple complexities.'' Which ones are appropriate depend on the questions being addressed. ''Disorder'' is not restricted to be the ratio of a nonequilibrium entropy to the corresponding equilibrium entropy; therefore, ''simple complexity'' need not vanish for all equilibrium systems, nor must it be nonvanishing for a nonequilibrium system.

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We are pleased that our contribution on a ''simple measure for complexity'' [1] (SDL) is of sufficient interest to have generated two comments, one by Crutchfield, Feldman, and Shalizi [2] (CFS) and another by Binder and Perry [3] (BP). We welcome the opportunity to respond and clarify our work.

In SDL we proposed

$$\Gamma_{\alpha\beta} \equiv \Delta^{\alpha}\Omega^{\beta}, \tag{1}$$

$$\Delta \equiv S/S_{\max}, \Omega \equiv 1-\Delta. \tag{2}$$

as a ''simple measure for complexity.'' α and β are (constant) parameters, S is the Boltzmann-Gibbs-Shannon entropy [4], and $S_{\max}$, the maximum entropy. Δ was introduced earlier by one of us as a measure for disorder, and Ω is referred to as ''order'' [5,6].

Let us first note that we have only a limited interest in terminology, and if someone does not like our use of the word ''complexity'' for the expression defined in Eq. (1), let them call it the ''Λ function'' or invent another term. The important thing is to have a clear definition of the terms one is using. For this reason, we will mostly refrain from calling $\Gamma_{\alpha\beta}$ ''complexity.''

CFS Point I: Since $S_{\max}$ is the equilibrium entropy, Δ and $\Gamma_{\alpha\beta}$ vanish for all equilibrium systems. This is a misinterpretation of SDL, perhaps due to our choice of a nonequilibrium system to illustrate the case where the entropy of the equiprobable distribution may not be the appropriate $S_{\max}$ and our statement that one *can* interpret $\Gamma_{\alpha\beta}$ as the product of ''order'' and ''distance from equilibrium.'' We did not write that ''$S_{\max}$ is taken to be the equilibrium entropy of the system . . . for *all* . . . systems.'' [2]. Neither ''disorder'' Δ nor $\Gamma_{\alpha\beta}$ is restricted to this interpretation. A perusal of our other work [5,7–9] will yield examples additional to those in SDL where $S_{\max}$ is not the equilibrium entropy of a nonequilibrium system. In fact, it is possible to have more than one $S_{\max}$, depending on the question(s) being addressed.

(a) Take the entropy of the universe as largely due to the black-body radiation background. The maximum conceivable entropy can be constructed by taking all the matter in the universe to make one black hole, yielding a very small ''disorder'' (see, e.g., Ref. [10]). In a sense that is an ultimate equilibrium entropy.

(b) The absolute maximum entropy possible is usually taken to be that of the equiprobable distribution.

(c) For a nonequilibrium system, one could take the entropy of the equilibrium system with the same number of particles, total energy, . . . , as the maximum entropy [1,11].

(d) The one-dimensional Ising system (two-state spins, only nearest neighbor interactions) provides a simple example where different $S_{\max}$'s are appropriate for answering different questions. The entropy is a function of the interaction parameter J, the external field B and temperature T: $S(B,J,T)$. The case of vanishing external field and vanishing interaction yields the equiprobable distribution and the absolute maximum entropy: $S(B=0,J=0,T)$ [12]. The absolute ''disorder,'' that with reference to $S(B=0,J=0,T)$, is then

$$\Delta = S(B,J,T)/S(B=0,J=0,T). \tag{3}$$

How much of the reduction of $S(B,J,T)$ compared to $S(B=0,J=0,T)$ is due to the interaction between spins? To answer this question we find the maximum of $S(B,J,T)$ with respect to J under the condition of constant net magnetization M (since, even for the paramagnet, the entropy varies with M). As expected, the entropy is maximum in the case of vanishing interaction, $J=0$. Thus, the maximum entropy under the constraint of constant net magnetization is

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$S(B_0,J=0,T)$, where B_0 is the value of the external field such that $M(B_0,J=0,T)=M(B,J,T)$. We now have a second ''disorder''

$$\Delta_{J=0}=S(B_0,J=0,T)/S(B=0,J=0,T). \tag{4}$$

This is the absolute ''disorder'' of the paramagnet with the same net magnetization as the Ising system with nonvanishing J. Since we have three entropies, $S(B,J,T)\leqslant S(B_0,J=0,T)\leqslant S(B=0,J=0,T)$, we can introduce a third ''disorder''

$$\hat{\Delta}=S(B,J,T)/S(B_0,J=0,T), \tag{5}$$

which is the ''disorder'' of the Ising system with respect to the paramagnet with the same net magnetization. The three ''disorders'' are related by $\Delta=\hat{\Delta}\Delta_{J=0}$.

The point is that there are many possible $S_{\max}$'s, and therefore ''disorders,'' ''orders,'' and ''complexities'' $\Gamma_{\alpha\beta}$, even for equilibrium systems. It is not in general true that Δ is identically 1 for equilibrium systems; therefore neither ''order'' nor ''complexity'' must vanish at equilibrium. When CFS write that as a consequence of $S_{\max}$'s being taken as the equilibrium entropy (for nonequilibrium systems) neither Δ nor $\Gamma_{\alpha\beta}$ can ''distinguish between two-dimensional Ising systems at low temperature, high temperature, or the critical temperature ... [n]or ... between the many different *kinds* of organization observed in equilibrium,'' this is an overly restrictive interpretation of Δ and $\Gamma_{\alpha\beta}$.

For a paramagnet ''pumped out of equilibrium,'' one could interpret S and $S_{\max}$ as the nonequilibrium entropy and the equilibrium entropy *under the appropriate constraints* [11], respectively. However, here again, CFS misinterpret our work. They wish to argue that since the pumped state is out of equilibrium, we would assign a nonzero level of complexity to this state. This is not true for this simple case of a paramagnet, for which the entropy can be written simply in terms of the total number of spins and the net magnetization. The key is to realize that the appropriate constraints here are just the total number of spins and the net magnetization; otherwise the nonequilibrium entropy could be greater than the equilibrium entropy. Since the total number of spins and the net magnetization must then be the same for the equilibrium and the nonequilibrium case, the entropies are the same, and we have maximum ''disorder'' and vanishing $\Gamma_{\alpha\beta}$. Incidentally, we have never maintained that $1-\Delta$, where Δ is the ratio of a nonequilibrium entropy to the corresponding equilibrium entropy, could distinguish different equilibrium distributions. To do this, one needs some of the various equilibrium disorders, as pointed out above.

CFS Point II. $\Gamma_{\alpha\beta}$ is ''overuniversal in the sense that it has the same dependence on disorder for structurally distinct systems.'' We assume they mean that $\Gamma_{\alpha\beta}$ always has the same dependence on ''disorder'' (given α and β). We agree with this as far as it goes; it follows from the definition of $\Gamma_{\alpha\beta}$. It is rather superficial though, and the question arises as to which $\Gamma_{\alpha\beta}$ and which ''disorder'' are meant. If $\Gamma_{\alpha\beta}$ is calculated from one ''disorder'' and its dependence on another ''disorder'' investigated, $\Gamma_{\alpha\beta}$ may well have a variable dependence on ''disorder.'' In Fig. 4 of SDL, we investigated $\Gamma_{11,J=0}$ as a function of Δ for one-dimensional Ising systems. This relation varies with J. Thus, it is not generally true that $\Gamma_{\alpha\beta}$ ''has the same dependence on disorder for structurally distinct systems.'' One has to be careful to clearly state which ''disorder'' and which ''complexity'' one is dealing with. If one does so, then $\Gamma_{\alpha\beta}$ may show different dependencies on ''disorder'' for ''structurally distinct systems'' and is not ''overuniversal'' in the sense used here.

Our calculation of $\Gamma_{11,J=0}$ as a function of Δ is analogous to Crutchfield and Feldman's [13] calculation of ''statistical complexity'' C_μ and ''excess entropy'' E, or ''effective measure complexity'' [14], again for one-dimensional Ising systems. In our interpretation of their results, they found C_μ, to within a multiplicative constant, to be $\Delta_{J=0}$ (we will return to this point below), and E, again to within a multiplicative constant, to be $\Delta_{J=0}-\Delta$. They then investigated the dependence of C_μ and E on Δ and found that these dependencies vary with J. From an ''order''-''disorder'' point of view, what they have investigated is the dependence of ''disorder'' under one set of conditions $(J=0)$ on ''disorder'' under another set of conditions $(J\neq 0)$, or in the case of E, the difference between these two ''disorders'' on one of the ''disorders.''

CFS Point III. The ''statistical complexity'' C_μ of one-dimensional spin systems [13] is not the same as the entropy of noninteracting spins. Therefore, the identification, to within a multiplicative constant, in SDL of C_μ with the ''disorder'' in the absence of interactions between spins is incorrect.

CFS have two objections, the first of which is dimensional inconsistency. This is not the place to get into a discussion of the proper units for entropy; let us just reiterate—to within a multiplicative constant. The second objection is that we ''conflate'' the definition of C_μ with Eq. (8) of Ref. [13], which is only valid within a limited range. Actually, we were not referring to that equation to identify C_μ, but rather to identify the excess entropy E. Be that as it may, our identification of C_μ with $\Delta_{J=0}$ in SDL is restricted to one-dimensional spin systems, which is what they treat in Ref. [13] and we treat in SDL. We were not ''conflating,'' but referring to their results for one-dimensional spin systems. They disagree with this, too, when they say that although $C_\mu=H(1)$, $H(1)$ is not the same as the entropy of noninteracting spins. However, on p. 1240 of Ref. [13] they write: ''For a NN system, Eq. (6) is equivalent to saying that $C_\mu=H(1)$, the entropy associated with the value of one spin.'' Earlier, p. 1239, they used the phrase ''isolated-spin uncertainty $H(1)$.'' Since an isolated spin can not be subject to interactions with neighboring spins, to our reading, they themselves have stated that $H(1)$ is the entropy of a spin subject to no interactions, and thus, to within a multiplicative constant, just $\Delta_{J=0}$. [Note that according to Ref. [13] the identification of C_μ with $H(1)$ breaks down for the paramagnetic case and the high temperature limit; we exclude these cases, too, of course.]

CFS Point IV. SDL mentions ''thermodynamic depth'' [15] as a complexity measure with a convex dependence on disorder, whereas Crutchfield and Shalizi [16] have shown that it is an increasing function of disorder. Our statement was based on the original exposition by Lloyd and Pagels [15] and other discussions (e.g., Ref. [17]). The results of Crutchfield and Shalizi [16] would indeed seem to indicate that thermodynamic depth is a monotonically increasing function of ''disorder,'' given their insistence that the choice of states to be made should be the ''causal states'' of ''ϵ machines.'' In particular, they object to ''judiciously redefining'' (p. 277) the ''appropriate set'' of macroscopic states. However, we believe the situation may not be so simple. First of all, they write [16] (p. 278): ''It is certainly not desirable to conflate a process's complexity with the complexity of whatever apparatus connects the process to the variables we happen to have seized upon as handles.'' This argument ignores the fact that the only access we have to real systems is through measurement. The situation would seem to be reminiscent of the endo-exophysics distinction (see, e.g., Ref [18]), at least superficially. Crutchfield and Shalizi take more of a endophysical point of view, while it would seem that Lloyd and Pagels take a more exophysical approach. Along a similar vein, the argument of Crutchfield and Shalizi seems to ignore the problem of frames of reference. For example, Andresen and Gordon [19] and Spirkl and Ries [20] have shown that a necessary condition for minimum entropy production in a continuous time system is a constant rate of entropy production in eigentime. In other words, for nonlinear systems the rate of entropy production will not be constant for an external observer, but only to the (nonlinear) system as it sees itself. In any case, ''thermodynamic depth'' would seem to be a convex measure of ''complexity'' in some cases. ''Back of the envelope'' calculations of ''thermodynamic depth,'' taken as the difference between a coarse-grained entropy and a fine-grained entropy [21], for a one-dimensional Ising ferromagnet indicate a convex dependence on ''disorder.'' However, this is not the case for an antiferromagnet with sufficiently negative J. Thus, ''thermodynamic depth'' may not qualify as either a convex or a monotonic complexity measure, or it may be either depending on the particulars of the system being investigated.

BP. $\Gamma_{\alpha\beta}$ does not capture all aspects of complexity; in particular, ''it does not describe the transition from regular to indexed languages observed at the period-doubling accumulation points of quadratic maps.'' We agree with Binder and Perry (BP) that results obtained on the basis of $\Gamma_{\alpha\beta}$ should be carefully interpreted and complemented with results based on other measures. However, we are not so sure that all of their statements are completely accurate. For example, they argue that ''effective measure complexity'' will ''[c]ertainly . . . pick up the nonregularity of a language,'' but that our measure will not in the logistic map. However, a comparison of Fig. 10 of [17] and Fig. 3 of SDL show that, as stated in SDL, ''major maxima as well as less major ones occur at the same values of r,'' although the relative values of the maxima differ.

Perhaps the main point of contention is that BP desire a ''complexity'' measure which can become infinite, whereas we purposefully constructed the measure in SDL so that it would not have this property (for $\alpha,\beta \geqslant 0$). Our reasoning is similar to that which argues that $S/S_{\max}$ is, for certain purposes, a ''better'' measure for''disorder'' than is the entropy S [5]. It would also seem that BP, like CFS, may have taken the definition of $\Gamma_{\alpha\beta}$ too literally in that they may not have realized that there may be several different $S_{\max}$'s, and therefore Δ's and $\Gamma_{\alpha\beta}$'s for a given system. Nonetheless, we would like to reiterate that we stated only that Γ_{11} behaves *similarly* to ''effective measure complexity'' for the logistic map, that it was not clear to us why this is so, and we do not know the breadth of systems for which this will be the case.

There is a plethora of proposed complexity measures in the literature, all trying to capture some aspects of what we mean when we say that something is complex. However, none of them capture all aspects of ''complexity.'' This is made explicit by the statement by CFS ''that a useful role for statistical complexity measures is to capture the structure—patterns, organization, regularities, symmetries—intrinsic to a process.'' We have nothing against this statement, unless one interprets ''a useful role'' as '' the only useful role,'' or one means that a measure of complexity must be a statistical complexity measure. There are many useful roles for complexity measures. [22] Perhaps at some time a consensus will arise; but until that time, we believe that there is a need for various approaches to complexity.

The situation becomes even more confused, when one realizes that even seemingly ''exact'' measures such as ''statistical complexity'' and ''effective measure complexity'' are not uniquely defined: ''For higher dimensional systems, e.g., spins in 2D, there are several ways to define E and C_μ.'' [13] (p. R1242). Thus, we believe there is a place for simple measures of complexity. The great advantage of $\Gamma_{\alpha\beta}$, as noted in SDL, is that it is available for systems where much less information is available than is necessary to calculate some other measures, such as E and C_μ.

We do not claim any ''universality'' for $\Gamma_{\alpha\beta}$ though, and think that one should examine several possible complexity measures to get a handle on the various things which can be meant by saying a system or a process is complex.

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